VIRTUAL WAITING TIME IN SINGLE-SERVER QUEUEING MODEL M|G|1 WITH UNRELIABLE SERVER AND CATASTROPHES

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In the present paper, the single-server queue model $M|G|1|\infty$ with unreliable server subject to catastrophes is considered. The transient and stationary distributions of virtual waiting time, busy period and idle state probability for two basic models with reliable and unreliable server are obtained.

Different generalizations of basic models are considered: model with batch arrival of customers, model with non-homogeneous streams of customers and catastrophes, model with k types of customers, model with k types of priority customers. For those models, the virtual waiting time distribution and idle state probability are found.

M|G|1 queuing models with catastrophes are widely used for modelling and performance evaluation in different fields such as:

- Computer and telecommunication systems and networks,
- Inventory and reliability systems,
- Biology, chemistry and physics,
- Manufacturing and transportation systems,
- Financial, and banking systems

Model Description

We consider a single server queueing model M|G|1|∞ with unreliable server and catastrophes. The arrival of customers and occurrence of catastrophes are according to a Poisson distribution with parameters λ and ν, respectively. Catastrophes cannot be served and accumulated in the model. They can just remove all the customers being in the model at the moment of their occurrence. Service times of customers are i.i.d. random variables β with general distribution function (PDF) $B(x)$ with density $b(x)$ and finite mean value b_1.

The server is unreliable and can break down and be repaired only when the model is free of customers. Time intervals between two consecutive server failures have exponential distribution with parameter α and the repair time have general distribution $\mathrm{C}(x)$ with density $\mathrm{c}(x)$ and finite mean value $\bar{\gamma}$. If the customers arrive during the server repair time, then they stay in the model, and after the server recovers the customers will be served.

The occurring catastrophe acts in the following manner: if the model is empty and the server is reliable, then it disappears without any influences on the model. If the server is busy serving customers, then the catastrophe removes all customers in the model, including the one in service. If the server is in repair, then the catastrophe interrupts the repair period and removes all customers in the model. After catastrophes, the model continues its work from an empty and reliable server state. All customers serve in the model according to FCFS (first come – first serve) discipline. The model has unlimited waiting space. The model is empty, the server is reliable and ready to serve customers at the initial moment $t = 0$.

Virtual Waiting Time

$\{\xi(t), t \geq 0\}$- is a stochastic process (SP) which describes workload or virtual waiting time of the model at the moment t and takes values in the state space $[0,\infty)$. For considering model SP $\xi(t)$, $t \geq 0$ is a homogeneous Markov process with respect to time. We will consider two different states of the SP $\xi(t)$: 0- state, where the model is free of customers and $(0, \infty)$ - where the model is busy either serving the customers or repairing the server.

$W(x,t) = P\{\xi(t) < x\}$ - is the probability distribution function for SP $\xi(t)$.

$p_0(t) = W(0+,t) = P\{\xi(t) = 0\}$ - is an idle state probability of the model, i.e. model is free from customers and the server is reliable, at the moment t .

$\widehat{W}(x,t) = W(x,t) - p_0(t) = P\{0 < \xi(t) < x\}$ - is a busy state probability of the model, i.e. the model is busy, at the moment t and total workload (virtual waiting time) of the model is less than x.

$$W(x,t) = \begin{cases} 0, & if \;\; x \leq 0 \\ \hat{W}(x,t) + p_0(t), & if \;\; x > 0, \end{cases}$$

$$W(\infty,t) = \hat{W}(\infty,t) + p_0(t) = 1 .$$

Virtual Waiting Time

By using standard probabilistic arguments for $\widehat{W}(x,t)$ and $p_0(t)$ we derive following equations

Case 1

$$\frac{d}{dt}p_0(t) = -\lambda p_0(t) + \frac{\partial}{\partial x}\hat{W}(x,t)\Big|_{x=0} + v\hat{W}(\infty,t),$$

$$\frac{\partial}{\partial t}\hat{W}(x,t) - \frac{\partial}{\partial x}\hat{W}(x,t) = -(\lambda+v)\hat{W}(x,t) - \frac{\partial}{\partial x}\hat{W}(x,t)\Big|_{x=0}$$

$$+\lambda\int_0^x \hat{W}(x-y,t)dB(y) + \lambda B(x)p_0(t),$$

Case 2

$$\frac{d}{dt}p_0(t) = -(\lambda+\alpha)p_0(t) + \frac{\partial}{\partial x}\hat{W}(x,t)\Big|_{x=0} + v\hat{W}(\infty),$$

$$\frac{\partial}{\partial t}\hat{W}(x,t) - \frac{\partial}{\partial x}\hat{W}(x,t) = -(\lambda+v)\hat{W}(x,t) - \frac{\partial}{\partial x}\hat{W}(x,t)\Big|_{x=0}$$

$$+\lambda\int_0^x \hat{W}(x-y,t)dB(y) + [\alpha C(x) + \lambda B(x)]p_0(t),$$

with initial conditions

$$W(\infty,t) = \hat{W}(\infty,t) + p_0(t) = 1, \quad and \quad W(0+,0) = p_0(0) = 1$$

Virtual Waiting Time

First we analyze the steady state solution of the model.

$$\hat{W}(x) = \lim_{t\to\infty} \hat{W}(x,t)\ ,\quad p_0 = \lim_{t\to\infty} p_0(t)$$

where p_0 - is a steady state probability of idle state of the model, and $\widehat{W}(x,t)$ is a PDF of workload of the model, respectively.

Case 1

$$0 = -\lambda p_0 + \frac{\partial}{\partial x}\hat{W}(x)\Big|_{x=0} + v\hat{W}(\infty)$$

$$\frac{\partial}{\partial x}\hat{W}(x) = (\lambda + v)\hat{W}(x) + \frac{\partial}{\partial x}\hat{W}(x)\Big|_{x=0} - \lambda\int_0^x \hat{W}(x-y)dB(y) - \lambda B(x)p_0$$

Case 2

$$0 = -(\lambda + \alpha)p_0 + \frac{\partial}{\partial x}\hat{W}(x)\Big|_{x=0} + v\hat{W}(\infty)$$

$$\frac{\partial}{\partial x}\hat{W}(x) = (\lambda + v)\hat{W}(x) + \frac{\partial}{\partial x}\hat{W}(x)\Big|_{x=0} - \lambda\int_0^x \hat{W}(x-y)dB(y) - [\alpha C(x) + \lambda B(x)]p_0$$

Virtual Waiting Time

Case 1 $$\frac{\partial}{\partial x}W(x)=(\lambda+v)W(x)-\lambda\int_0^x W(x-y)dB(y)-v$$

Case 2 $$\frac{\partial}{\partial x}W(x)=(\lambda+v)W(x)-\lambda\int_0^x W(x-y)dB(y)-v+\alpha[1-C(x)]p_0$$

Let $\widetilde{W}(s)$ be the Laplace - Stieltjes transformation (LST) of a function $W(x)$

$$\tilde{W}(s)=\int_0^\infty e^{-sx}dW(x).$$

Case 1 $$\tilde{W}(s)=\frac{sp_0-v}{s-v-\lambda(1-\tilde{B}(s))},$$

Case 2 $$\tilde{W}(s)=\frac{[s+\alpha(1-\tilde{C}(s))]p_0-v}{s-v-\lambda(1-\tilde{B}(s))}.$$

Virtual Waiting Time

Case 1

$$\tilde{\hat{W}}(s) = \tilde{W}(s) - p_0 = \frac{p_0\lambda\bar{\tilde{\bar{B}}}(s) - v(1-p_0)}{s - v - \lambda\bar{\tilde{\bar{B}}}(s)} = \frac{p_0\lambda\bar{\tilde{\bar{B}}}(s) - v\hat{W}(\infty)}{s - v - \lambda\bar{\tilde{\bar{B}}}(s))}.$$

Case 2

$$\tilde{\hat{W}}(s) = \frac{p_0[\alpha\bar{\tilde{\bar{C}}}(s) + \lambda\bar{\tilde{\bar{B}}}(s)] - v(1-p_0)}{s - v - \lambda\bar{\tilde{\bar{B}}}(s)} = \frac{p_0[\alpha\bar{\tilde{\bar{C}}}(s) + \lambda\bar{\tilde{\bar{B}}}(s)] - v\hat{W}(\infty)}{s - v - \lambda\bar{\tilde{\bar{B}}}(s)}$$

where $\bar{A}(x) = 1 - A(x)$, and $\hat{W}(\infty) = 1 - p_0$.

The idle state probabilities

Case 1 $\quad p_0 = \dfrac{1}{1+\lambda\hat{\bar{\pi}}_1}, \quad p_1 = \dfrac{\lambda\hat{\bar{\pi}}_1}{1+\lambda\hat{\bar{\pi}}_1}.$

Case 2 $\quad p_0 = \dfrac{1}{1+\hat{\bar{\pi}}_1(\lambda+\alpha)}, \quad p_1 = \dfrac{\hat{\bar{\pi}}_1(\lambda+\alpha)}{1+\hat{\bar{\pi}}_1(\lambda+\alpha)}$

Model Analysis

$\tilde{\pi}_1(s)$- is LST of busy period of standard M|G|1 model

$$\tilde{\pi}_1(s) = \tilde{B}(s + \lambda(1 - \tilde{\pi}_1(s)), \quad \mathrm{Re}\, s > 0$$

$\tilde{\pi}_2(s)$- is LST of busy period of standard M|G|1 model with unreliable server

$$\tilde{\pi}_2(s) = \frac{\lambda}{\lambda + \alpha}\tilde{B}(s + (\lambda + \alpha)(1 - \tilde{\pi}_2(s)) + \frac{\alpha}{\lambda + \alpha}\tilde{C}(s + (\lambda + \alpha)(1 - \tilde{\pi}_2(s)) \quad \mathrm{Re}\, s > 0.$$

LST and mean value of busy period of standard M|G|1 model with catastrophes

$$\tilde{\tilde{\pi}}_i(s) = \tilde{\pi}_i(s + v) + \frac{v}{s + v}[1 - \tilde{\pi}_i(s + v)] = \frac{v + s\tilde{\pi}_i(s + v)}{s + v}, \; \hat{\bar{\pi}}_{1i} = \frac{1 - \tilde{\pi}_i(v)}{v}, \quad i = 1, 2.$$

Model Analysis

The steady-state probabilities p_0 and p_1

Case 1 $$p_0 = \frac{v}{v+\lambda[1-\tilde{\pi}_1(v)]}, \quad p_1 = \frac{\lambda[1-\tilde{\pi}_1(v)]}{v+\lambda[1-\tilde{\pi}_1(v)]},$$

Case 2 $$p_0 = \frac{v}{v+[1-\tilde{\pi}_2(v)](\lambda+\alpha)}, \quad p_1 = \frac{[1-\tilde{\pi}_2(v)](\lambda+\alpha)}{v+[1-\tilde{\pi}_2(v)](\lambda+\alpha)}.$$

LST of virtual waiting time distribution W(s) for the corresponding models M|G|1 with catastrophes

Case 1 $$\tilde{W}(s) = \frac{v[s-v-a(1-\hat{\tilde{\pi}}(v))]}{[s-v-\lambda\bar{B}(s)][v+a(1-\hat{\tilde{\pi}}(v))]}.$$

Case 2 $$\tilde{W}(s) = \frac{v[s-v(1-\tilde{C}(s))]\{1-\tilde{C}(s)[v+\lambda[1-\tilde{\pi}(v)](1+\alpha v)]\}}{[s-v-\lambda(1-\tilde{B}(s))][v+\lambda[1-\tilde{\pi}(v)](1+\alpha v)]}.$$

Model Analysis

Let $\bar{\omega}_1$ be a mean value of virtual waiting time for the model M|G|1 with reliable server

$$\bar{\omega}_1 = \lim_{s\to 0}\left[-W'(s)\right] = \frac{p_0 - [1-\lambda b_1]}{\nu} = \frac{p_0 - [1-\rho]}{\nu},$$

where ρ is a workload of the model, $\rho = \lambda b_1$.

$$\bar{\omega}_1 = \frac{p_0 - (1-\rho)}{\nu} \cong \frac{1}{2}\frac{\lambda b_2}{(1-\rho)} + \nu\left(\frac{1}{6}\frac{\lambda b_3}{(1-\rho)^2} + \frac{3}{4}\frac{(\lambda b_2)^2}{(1-\rho)^3}\right) + O(\nu^2)$$

$$p_0 = \hat{p}_0 + \frac{\nu}{2}\frac{\lambda b_2}{(1-\rho)} + \nu^2\left(\frac{1}{6}\frac{\lambda b_3}{(1-\rho)^2} + \frac{3}{4}\frac{(\lambda b_2)^2}{(1-\rho)^3}\right) + O(\nu^3),$$

$$\bar{\omega}_1 = \hat{\bar{\omega}}_1 + \nu\left(\frac{1}{6}\frac{\lambda b_3}{(1-\rho)^2} + \frac{3}{4}\frac{(\lambda b_2)^2}{(1-\rho)^3}\right) + O(\nu^2).$$

Where $\hat{\bar{\omega}}$ and $\hat{\bar{p}}$ are the mean value of virtual waiting time and idle state stationary probability for the standard model M|G|1 without catastrophes.

Model Analysis

Transient virtual waiting time of the model M|G|1

Equations for LST of virtual waiting time distribution W(*x*,*t*)

Case 1 $$\frac{\partial}{\partial t}\tilde{W}(s,t)-\tilde{W}(s,t)[s-v-\lambda(1-\tilde{B}(s))]=v-sp_0(t)\ .$$

Case 2 $$\frac{\partial}{\partial t}\tilde{W}(s,t)-\tilde{W}(s,t)[s-v-\lambda(1-\tilde{B}(s))]=v-p_0(t)[s+\alpha\bar{\tilde{C}}(s)].$$

and their solutions

Case 1 $$\tilde{W}(s,t)=e^{[s-v-\lambda\bar{\tilde{B}}(s)]t}\left\{\tilde{W}(s,0)-s\int_0^t e^{-[s-v-\lambda\bar{\tilde{B}}(s)]u}p_0(u)du+v\int_0^t e^{-[s-v-\lambda\bar{\tilde{B}}(s)]u}du\right\}$$

Case 2

$$\tilde{W}(s,t)=e^{[s-v-\lambda\bar{\tilde{B}}(s)]t}\left\{\tilde{W}(s,0)-[\alpha\bar{\tilde{C}}(s)+s]\int_0^t e^{-[s-v-\lambda\bar{\tilde{B}}(s)]u}p_0(u)du+v\int_0^t e^{-[s-v-\lambda\bar{\tilde{B}}(s)]u}du\right\}.$$

Model Analysis

The LT of idle state probabilities

$$\tilde{p}_0(s) = \frac{s+v}{s\left\{s+v+\lambda(1-\tilde{\tilde{\pi}}(s+v))\right\}}$$

$$\tilde{p}_0(s) = \frac{s+v}{s\left\{s+v+\lambda(1-\tilde{\tilde{\pi}}(s+v))+\alpha[1-\tilde{C}(s+v+\lambda(1-\tilde{\tilde{\pi}}(s+v)))]\right\}}.$$

Remark **1**.

Let's suppose that customers arrive and catastrophes occur according to non-homogeneous Poisson processes with rates $\lambda(t)$ and $v(t)$, respectively.
In this case the corresponding differential equations and their solutions for $W(s,t)$ are

Case 1 $$\frac{\partial}{\partial t}\tilde{W}(s,t)-\tilde{W}(s,t)[s-v(t)-\lambda(t)\tilde{\bar{B}}(s)]=v(t)-sp_0(t),$$

Case 2 $$\frac{\partial}{\partial t}\tilde{W}(s,t)-\tilde{W}(s,t)[s-v(t)-\lambda(t)\tilde{\bar{B}}(s)]=v(t)-p_0(t)[s+\alpha\tilde{\bar{C}}(s)].$$

Case 1
$$\tilde{W}(s,t)=e^{\int_0^t[s-v(y)-\lambda(y)\bar{\tilde{B}}(s)]dy}\times\left\{\tilde{W}(s,0)-s\int_0^t e^{\int_0^u[s-v(y)-\lambda(y)\bar{\tilde{B}}(s)]dy}p_0(u)du+\int_0^t e^{\int_0^u[s-v(y)-\lambda(y)\bar{\tilde{B}}(s)]dy}v(u)du\right\}$$

Case 2
$$\tilde{W}(s,t)=e^{\int_0^t[s-v(y)-\lambda(y)\bar{\tilde{B}}(s)]dy}\times\left\{\tilde{W}(s,0)-[\alpha\tilde{\bar{C}}(s)+s]\int_0^t e^{\int_0^u[s-v(y)-\lambda(y)\bar{\tilde{B}}(s)]dy}p_0(u)du+\int_0^t e^{\int_0^u[s-v(y)-\lambda(y)\bar{\tilde{B}}(s)]dy}v(u)du\right\}.$$

***Remark* 2.**

Let's consider the model BM|G|1 with batch arrival of customers, catastrophes and unreliable server. The batches arrive according to Poisson distribution with parameter λ. With probability r q arriving batch contains r customers. Let $Q(z)$ be the probability generating function (PGF) of batch size and q is the mean value of the batch size.

$$Q(z) = \sum_{r=0}^{\infty} q_r z^r .$$

Then for the LST of virtual waiting time distribution of the model we derive

$$\tilde{W}(s,t) = e^{\varphi(s,v)t} \left\{ \tilde{W}(s,0) - [\alpha \bar{\tilde{C}}(s) + s] \int_0^t e^{\varphi(s,v)u} p_0(u)du + \int_0^t e^{\varphi(s,v)u} v du \right\}$$

where $\varphi(s,v) = s - v - \lambda Q(\tilde{B}(s))$.

***Remark* 3.**

Let's consider queuing model M|G|1 with catastrophes when arrival rate and PDF of service time of first customer which opens the busy period are λ_0 and $B_0\,(x)$ and the arrival rate and PDF of other customers (during one busy period of the model) are λ_1 and $B_1\,(x)$, respectively. This model can be used for modeling queuing systems with unreliable server, vacations and set up time.
We suppose that both PDFs have finite first and second moments.
The corresponding differential equations for $W(x,t)$, LST $\tilde{W}(s,t)$ and its solution are

$$\frac{\partial}{\partial t}W(x,t)-\frac{\partial}{\partial x}W(x,t)=-(\lambda+v)W(x,t)+\lambda\int_0^x W(x-y,t)dB(y)$$
$$+[\lambda_0 B_0(x)-\lambda B(x)-\alpha\bar{C}(x)]p_0(t),$$

$$\frac{\partial}{\partial t}\tilde{W}(s,t)-\tilde{W}(s,t)[s-v-\lambda\tilde{\bar{B}}(s)]=v-p_0(t)[s+\alpha\tilde{\bar{C}}(s)-\lambda_0\tilde{B}_0(s)+\lambda\tilde{B}(s)],$$

$$\tilde{W}(s,t)=e^{[s-v-\lambda\tilde{\bar{B}}(s)]t}\times$$
$$\left\{\tilde{W}(s,0)-[s+\alpha\tilde{\bar{C}}(s)-\lambda_0\tilde{B}_0(s)+\lambda\tilde{B}(s)]\int_0^t e^{-[s-v-\lambda\tilde{\bar{B}}(s)]u}p_0(u)du+v\int_0^t e^{-[s-v-\lambda\tilde{\bar{B}}(s)]u}du\right\}.$$

Thank you